Answers to selected problems

Derivatives and Integrals

1. The easiest approach is to note that the discontinuities in slope occur at $x = 1$ and $x = 2$ so the function must have an equation of the form

$$f(x) = \alpha|x-1| + \beta|x-2| + \gamma$$

where α, β and γ are to be found. For $x > 2$

$$\alpha x - \alpha + \beta x - 2\beta + \gamma = -5x + 11$$

so, equating cofficients, $\alpha + \beta = -5$ and $-\alpha - 2\beta + \gamma = 11$. Proceeding similarly for $1 < x < 2$ and $x < 1$ gives a further set of equations for α, β and γ which can be solved to give $\alpha = -6$, $\beta = 1$ and $\gamma = 7$ and $f(x) = -6|x-1| + |x-2| + 7$.We can then check, for example, that for $x < 1$, $f(x) = -6 + 6x + 2 - x + 7 = 5x + 3$ as required.

3. We have $y(x) = xe^x$ and hence $y'(x) = e^x + xe^x$, $y''(x) = 2e^x + xe^x$ and $y'''(x) = 3e^x + xe^x$.Thus we guess $y^{(n)} = ne^x + xe^x$ as is easily checked by induction.

The formula for the n-th derivative of the product uv must be valid for $u = x$ and $v = e^x$, so $a = n$.
If $z = x\sin(x)$ the formula gives $(x\sin(x))^{(8)} = x(\sin(x))^{(8)} + 8(\sin(x))^{(7)}$.
At $x = 0$ this is $8(-1)^3\cos(0) = -8$.

5. Completing the square in the denominator gives $1 + x + x^2 = (x + \frac{1}{2})^2 + \frac{3}{4}$ and letting $y = \sqrt{3}/2$ the integral becomes

$$\begin{aligned} \frac{2}{\sqrt{3}} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dy}{y^2+1} &= \frac{2}{\sqrt{3}} \left[\tan^{-1}(y)\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}. \end{aligned}$$

Also, since $(1-x)(x^2+x+1) = (1-x^3)$

$$\frac{1}{1+x+x^3} = \frac{1-x}{1-x^3} = (1-x)(1+x^3+x^6+\ldots).$$

Multiplying out and integrating term by term gives

$$\begin{aligned} \int_0^1 \frac{dx}{1+x+x^2} &= \left[x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + \cdots - x - \frac{1}{2}x^2 - \frac{1}{5}x^5 - \frac{1}{8}x^8 \ldots\right]_0^1 \\ &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \ldots \end{aligned}$$

from which the result follows.

7.(a) If

$$f(t) = \frac{t^2}{(1+t)(2+t)^2} = \frac{A}{1+t)} + \frac{B}{(2+t)^2}$$

then

$$t^2 = A(2+t)^2 + B(1+t) = A(4+4t+t^2) + B(1-t).$$

Comparing coefficients gives $A = 1$ and $B = -4$.

(b) Integrating $f(t)$ between 0 and x gives

$$\int_0^x f(t)dt = \left[\ln(1+t) + 4(2+t)^{-1}\right]_0^x = \ln(1+x) - \frac{x}{1+x/2}$$

provided that the argument of the logarithm is positive, i.e. provided $x > -1$.

(c) Put $x = 2/5$ to obtain $\ln(\frac{7}{5}) > \frac{2/5}{6/5} = 1/3$ and hence, exponentiating, $e < (7/5)^3$.

(d) (i)

$$\begin{aligned} f(x) =& \frac{1}{1+x} - \frac{4}{(2+x)^2} \\ =& 1 - x + x^2 - x^3 + \ldots - \left[1 - x + \frac{(-2)(-3)}{2}\left(\frac{x}{2}\right)^2 + \ldots\right] \\ =& \frac{x^2}{4} - \frac{x^3}{2} + \ldots \end{aligned}$$

valid for $|x| < 1$ (ii) From the identity proved in part (b) the simplest appraoch ois to integrate the preceeding series:

$$\ln(1+x) - \frac{x}{1+x/2} = \int_0^x f(t)dt = \frac{x^3}{12} - \frac{x^4}{8} + \ldots$$

also valid for $|x| < 1$.

9. We have $J_0 = \displaystyle\int_0^{\pi/2} d\theta = \pi/2$ and $J_1 = \displaystyle\int_0^{\pi/2} \cos(\theta)d\theta = 1$. Then, integrating by parts,

$$\begin{aligned} J_m &= \int_0^{\pi/2} \cos^{(m-1)}(\theta) d\sin(\theta) \\ &= \left[\sin(\theta)\cos^{(m-1)}(\theta)\right]_0^{\pi/2} + \int_0^{\pi/2} (m-1)\cos^{(m-1)}(\theta)\sin^2(\theta)d\theta \end{aligned}$$

Replacing $\sin^2(\theta)$ with $1 - \cos^2(\theta)$ and re-arranging, we get

$$J_m = \frac{(m-1)}{m} J_{m-2}.$$

Hence

$$J_{2n} = \frac{2n-1}{2n}\frac{2n-3}{2n-2}\cdots\frac{1}{2}J_0 \quad \text{and} \quad J_{2n+1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}J_1.$$

We can therefore findJ_{2n}/J_{2n+1}, using the values of J_0 and J_1:

$$\begin{aligned}\frac{J_{2n}}{J_{2n+1}} &= \frac{2n-1}{2n}\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{2n+1}{2n}\frac{2n-1}{2n-2}\cdots\frac{3}{2} \\ &= \left(\frac{3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2 (2n+1)\frac{\pi}{2}\end{aligned}$$

which gives nthe required expression for $\pi/2$ on rearrangement. Now, since $\cos(\theta) \leq 1$ we have

$$J_{2n+1} = \int_0^{\pi/2} \cos^{2n+1}(\theta)d\theta \leq \int_0^{\pi/2} \cos^{2n}(\theta)d\theta = J_{2n}$$

and, continuing, that $J_{2n+1} \leq J_{2n} \leq J_{2n-1}$. Dividing by J_{2n+1}, it follows that

$$\frac{J_{2n+1}}{J_{2n+1}} \leq \frac{J_{2n}}{J_{2n+1}} \leq \frac{J_{2n-1}}{J_{2n+1}} = \frac{2n+1}{2n}.$$

Taking the limit as $n \to \infty$ gives

$$1 \leq \lim_{n\to\infty} \frac{J_{2n}}{J_{2n+1}} \leq \lim_{n\to\infty} \frac{J_{2n-1}}{J_{2n+1}} \leq 1$$

and hence that both limits tend to 1. The final result follows by taking the limit of the expression for $\pi/2$ and slightly rearranging the bracket.

Elementary Functions

1.
$$\frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] = \frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right) = \frac{1}{1-x^2}.$$

Also

$$\begin{aligned}\int 1.\tanh^{-1}(x)dx &= x\tanh^{-1}(x) - \int \frac{x}{1-x^2}dx \\ &= x\tanh^{-1}(x) + \frac{1}{2}\ln(1-x^2) + c.\end{aligned}$$

3. Let $\sinh^{-1}(x) = y$. Then $x = \sinh(y)$ and $1 = \cosh(y)\frac{dy}{dx}$ so $\frac{dy}{dx} = \frac{1}{\sqrt{1+y^2}}$.

Also,

$$\begin{aligned}\int 1.\sinh^{-1}(x)dx &= x\sinh^{-1}(x) - \int \tfrac{x}{\sqrt{1+x^2}}dx \\ &= x\sinh(x) - \sqrt{1+x^2} + c.\end{aligned}$$

Functions, limits and series

1. The denominator of each term has a factor $(x+2)$ so we have

$$\begin{aligned} f(x) &= \frac{1}{(x+2)}\left[\frac{x^3+4x^2+Ax+1}{x} - \frac{x^3+x^2-4}{(x-2)}\right] \\ &= \frac{x^3+(A-8)x^2+(5-2A)x-2}{x(x-2)}\end{aligned}$$

For the expression to have a finite limit the numerator must have a factor (+2) (or, equivalently, equal 0 for $x=-2$). Factorising

$$x^3+(A-8)x+(5-2A)x-2 = (x+2)(x^2+(A-10)x+1)$$

provided that (comparing the terms in x) $5 - 2A = 2(A - 10) - 1$ or $A = 13/2$. Then

$$f(x)$$

which tends to 0 as $x \to -2$. From the simplified expression for $f(x)$ we see that as $x \to 0^+$, $f(x) \to -1/(-2x) \to +\infty$; as $x \to 0^-$, $f(x) \to -\infty$; as $x \to 1$, $f(x) \to (3/2)/(-1) = -3/2$; and as $x \to \pm\infty$, $f(x)\ x^2/x^2 = 1$.

3. (a)Dividing the numerator by the denominator (and factorising the denominator) gives

$$\begin{aligned} f(z) &= 3z + \frac{-6z^2 + 10z - 2}{z(z-2)(z-1)} \\ &= 3z + \frac{a}{z} + \frac{b}{z-2} + \frac{c}{z-1}. \end{aligned}$$

Cross-multiplying and comparing coefficients (or any other method forfinding partial fractions) gives

$$(a+b+c)z^2 - (3a+b+2c)z + 2a = -6z^2 + 10z - 2$$

from which $a = -1$, $b = -3$ and $c = -2$.

(b)

$$\begin{aligned} f(z) &= 3z - \frac{1}{z} + 2(1 + z + z^2 + \ldots) + \frac{3}{2}(\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \ldots\right) \\ &= -\frac{1}{z} + \frac{7}{2} + \frac{23}{4}z + \ldots. \end{aligned}$$

(c)

$$\begin{aligned} \int_a^b f(z)dz &= \left[\frac{3}{2}z^2 - \ln(z) - 3\ln|z-2| - 2\ln|z-1|\right]_a^b \\ &= \frac{3}{2}(b^2 - a^2) - \ln(\frac{b}{a}) - 3\ln\left(\frac{1-b/2}{1-a/2}\right) - 2\ln\left(\frac{1-b}{1-a}\right). \end{aligned}$$

If $2 > b > 1$ (and $a < 1$) the term $(1 - b)$ is replaced by $b - 1$; if $a < 1$ and $b > 2$ in addition, $(1 - b/2)$ is replaced by $(b/2 - 1)$; if $2 > a > 1$ and $2 > b > 1$ then $1 - a$ is replaced by $a - 1$ also.

5. NOT DONE

7. We have $\cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \ldots$ and hence, $\cos(\theta)\sec(\theta) = 1$ gives

(a)

$$\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \ldots\right)\left(a_0 + a_1\theta + a_2\theta^2 + \ldots\right).$$

Comparing coefficients, we get $a_0 = 0$, $a_1 = 0$, $a_2 = a_1/2 = 1/2$, $a_3 = 0$, $a_4 = a_2/2 - a_0/24 = 1/4 - 1/24 = 5/24$, $a_6 = a_4/2 - a_2/4! + a_0/6! = 5/48 - 1/48 + 1/(30 \times 24) = 61/720$.

(b) Hence, $\sec(0) = 1$, and differentiating the series and putting $\theta = 0$, $\sec^{(2)}(0) = 2a_2 = 1$, $\sec^{(4)}(0) = (4!)a_4 = 5$ and $\sec^{(6)}(0) = (6!)a_6 = 61$.

(c) From the series

$$\lim_{\theta \to 0}\left[\frac{\sec(\theta) - 1}{\theta^2}\right] = 1/2$$

and from l'Hôpital's rule,

$$\lim_{\theta \to 0}\left[\frac{\sec(\theta) - 1}{\theta^2}\right] = \lim_{\theta \to 0}\left[\frac{\sec(\theta)\tan(\theta)}{2\theta}\right] = \lim_{\theta \to 0}\left[\frac{\sec^2(\theta)}{2}\frac{\sin(\theta)}{\theta}\right] = \frac{1}{2}$$

9. (i) From $(1 - x)^{-1} = 1 + x + x^2 + \ldots$, we get

$$\int \frac{dx}{1 - x} = -\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots.$$

(ii) Obviously $-\ln(1 - x) > x + x^2/2$ because we are omitting positive terms. Also

$$\begin{aligned} -\ln(1-x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3}\left(1 + \frac{3x}{4} + \frac{3x^2}{5} + \dots\right] \\ &< x + \frac{x^2}{2} + \frac{x^3}{3}\left(1 + x + x^2 + \dots\right) \\ &= x + \frac{x^2}{2} + \frac{x^3}{3(1-x)} \end{aligned}$$

Let $x = 1/10$ then

$$\ln(\frac{10}{9} = -\ln(\frac{9}{10} = -\ln(1 - \frac{1}{10}) > \frac{1}{10} + \frac{1}{200} = \frac{21}{200}$$

by less than $\dfrac{x^3}{3(1-x)} = \dfrac{1}{2700}$ or $\dfrac{1}{2700}/\dfrac{21}{200} \times 100$ % or about $\frac{1}{3}$ %.

(iii) From the series

$$\lim_{x\to 0} \frac{\ln(1-x) - x}{x} = \lim_{x\to 0} \frac{x^2/2}{x} = 0,$$

and from l'Hôpital's rule,

$$\lim_{x\to 0} \frac{\frac{1}{1-x} - 1}{1} = 0.$$

11. (a) $\displaystyle\int (x+1)e^x dx = [(x+1)e^x] - \int e^x dx = xe^x + c.$
Either similarly, or putting $x = -y$ gives
$\displaystyle\int (x-1)e^{-x} dx = \int (y+1)e^y dy = ye^y + c = -xe^{-x} + c.$

(b) At $x = 2$, $2e^{-2} = 3Ae^2$, so $A = \frac{1}{3}e^{-4}$.

(c) $\int_2^X f(x)dx = \left[-xe^{-x}\right]_2^X = 2e^{-2} - Xe^{-X}$.
$\int_2^\infty f(x)dx = \lim_{X\to\infty} \int_2^X f(x)dx = 2e^{-2}$.

(d) $\int_Y^2 f(x)dx = [Axe^x]_Y^2 = 2Ae^2 - YAe^Y \to 2Ae^2$ (as $Y \to -\infty$) $= \frac{2}{3}e^{-2}$.
Hence $\int_{-\infty}^\infty f(x)dx = (2 + \frac{2}{3})e^{-2} = \frac{8}{3}e^{-2}$.

Vectors

1. The direction of the line is $\mathbf{t} = \mathbf{d} - \mathbf{c}$ so the line is $\mathbf{r} = \mathbf{c} + \lambda\mathbf{t}$.

DIAGRAM

For a point $\mathbf{r}$ in the plane, $\mathbf{r}-\mathbf{a}$ is in nthe plane and hence perpendicular to the normal to the plane $\mathbf{n}$. Thus $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

At the point of intersection, $\mathbf{r} = \mathbf{c} + \lambda\mathbf{t}$ and $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$. Therefore $(\mathbf{c} + \lambda\mathbf{t}) \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ from which $\lambda = [(\mathbf{a} - \mathbf{c}) \cdot \mathbf{n}]/\mathbf{t} \cdot \mathbf{n}$. Hence the pin tof intersection is

$$\mathbf{r} = \mathbf{c} + \frac{(\mathbf{a} - \mathbf{c}) \cdot \mathbf{n}}{\mathbf{t} \cdot \mathbf{n}}\mathbf{t}.$$

If $\mathbf{a} = \mathbf{d}$ then $\lambda = 1$ and the intersection is at $\mathbf{r} = \mathbf{c} + \mathbf{t} = \mathbf{d}$.

3 (ii) Interchanging "dot and cross": $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{d})) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$.

$[\mathbf{a} \times \mathbf{b}] \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a} \times \mathbf{b}] \cdot \mathbf{d}\mathbf{c} - [\mathbf{a} \times \mathbf{b}] \cdot \mathbf{c}\mathbf{d}$ or, in temrs of the scalar triple product, $[\mathbf{a}, \mathbf{b}\mathbf{d}]\mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d}$. (Equivalently $[\mathbf{b}, \mathbf{c}, \mathbf{d}]\mathbf{a} - [\mathbf{a}, \mathbf{c}, \mathbf{d}]\mathbf{b}$.)

5. For simplicity we put the origin at O. Then $-\mathbf{a} + \mathbf{k} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$ from which $\mathbf{a} - \mathbf{b} = 2\mathbf{a} - 2\mathbf{k}$. The diameter is $|\mathbf{b} - \mathbf{a}| = 2|\mathbf{a} - \mathbf{k}| = 2[(\mathbf{a} - \mathbf{k}) \cdot$

$(\mathbf{a}-\mathbf{k}))]^{1/2} = 2(a^2 - 2\mathbf{a}\cdot\mathbf{k} + k^2)^{1/2} = 2(4-2+1)^{1/2} = 2\sqrt{3}$.

7. (i) $\mathbf{r} = \mathbf{c} + \lambda\mathbf{a} + \mu\mathbf{b}$

(ii) Taking the scalar product with $\mathbf{a}\times\mathbf{b}$ gives $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$. Putting in the given values we get $\mathbf{a}\times\mathbf{b} = (-11, 7, 1)$ and $-11x + 7y + z = 4$ as the equation of the plane.

(iii) The line is $\mathbf{r} = \mathbf{c} + \nu(\mathbf{a}\times\mathbf{b})$ or $(x, y, z) = (1 - 11\nu, 2 + 7\nu, 1 + \nu)$. This cuts the (x, y) plane when $1 + \nu = 0$) i.e. at $\mathbf{d} = (12, -5, 0)$.

(iv) The $x-$axis is $y = 0$, $z = 0$ so the plane cuts the $x-$axis at $x = -4/11$ so $\mathbf{m}_1 = (-4/11, 0, 0)$. Similarly $\mathbf{m}_2 = (0, 4/7, 0)$ and $\mathbf{m}_3 = (0, 0, 4)$.

(v) The sides of the tetrahedron are $\mathbf{s}_1 = \mathbf{d} - \mathbf{m}_1 = (\frac{136}{11}, -5, 0)$, $\mathbf{s}_2 = \mathbf{d} - \mathbf{m}_2 = (12, -\frac{39}{7}, 0)$ and $\mathbf{s}_3 = \mathbf{d} - \mathbf{m}_3 = (12, -5, -4)$. The volume of the tetrahedron ($\frac{1}{3}$ base $\times$ height) $= \frac{1}{3}(\mathbf{s}_1\cdot\mathbf{s}_2\times\mathbf{s}_3 = 912/77$.

9. The unit normal $\mathbf{n}$ to the plane $\mathbf{n}\cdot\mathbf{r} = p$ is here $\mathbf{n} = (3, -1, 1)/\sqrt{11}$. The angle θ between $\mathbf{n}$ and $\mathbf{a}$ is given by $\cos(\theta) = \mathbf{n}\cdot\mathbf{p}/|\mathbf{n}||\mathbf{p}| = (3 + 2)/\sqrt{11}\sqrt{5} = \sqrt{(}5/11$. Thus $\theta = 47.6^\circ$. The line $\mathbf{r} = \lambda\mathbf{a}$ cuts the plane at $\lambda\mathbf{a}\cdot\mathbf{n} = p$ i.e. at $\lambda = 1/\cos(theta)$ or $\mathbf{r} = \sqrt{11/5}(1, 0, 2)$.

11. The line is $\mathbf{r} = \sqrt{3}/2\mathbf{k} + \lambda/\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ so point at unit distance from O is given by $|\mathbf{r}| = 1$ or $(\lambda^2/3 + \lambda^2/3 + (\lambda/\sqrt{3} + \sqrt{3}/2)^2) = 1$ from which $\lambda = -3/2$ or $1/2$. Taking $\lambda = 1/2$ the line l_2 is $\mathbf{r} = \mu\mathbf{b} = \mu(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} = \frac{\mu}{2\sqrt{3}}(1, 1, 4)$. The normal to the required plane is perpendicular to l_1 and l_2 hence in the direction $\mathbf{n} = \mathbf{t}\times\mathbf{b} = (3, -3, 0)$. The plane is therefore $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ or $3x - 3y = 0$.

Matrices

1.

$$\det\begin{pmatrix}1-\lambda & 3 & 0\\ 3 & -2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{pmatrix}=0$$

if

$$(1-\lambda)[(-2-\lambda)(1-\lambda)-1]-3(1-\lambda)=0$$

so

$$\lambda=1 \qquad \text{or} \qquad \lambda^2+\lambda-12=0$$

hence if $\lambda=1$, $\lambda=3$ or $\lambda=-4$. To find the eigenvectors $\mathbf{x}$ we solve the simultaneous equations $(\mathsf{A}-\lambda\mathsf{I})\mathbf{x}=0$ for these values of λ in turn. The corresponding eigenvectors are $k(1,0,3)$, $k(-3,-2,1)$ and $k(-3,5,1)$. We can check that the eigenvectors are mutually orthogonal,. For example: $(1,0,3)\cdot(-3,5,-1)=0$. Now

$$\begin{aligned}
&\begin{pmatrix}1 & 0 & 3\\ -3 & -2 & 1\\ -3 & 5 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 0\\ 3 & -2 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & -3 & -3\\ 0 & -2 & 5\\ 3 & 1 & 1\end{pmatrix}\\
&=\begin{pmatrix}1 & 0 & 3\\ -3 & -2 & 1\\ -3 & 5 & 1\end{pmatrix}\begin{pmatrix}1 & -9 & 12\\ 0 & -6 & -20\\ 3 & 3 & -4\end{pmatrix}\\
&=\begin{pmatrix}10 & 0 & 0\\ 0 & 48 & 0\\ 0 & 0 & -140\end{pmatrix}
\end{aligned}$$

3. Let

$$\mathsf{A}=\begin{pmatrix}1 & 3 & 2\\ 2 & 1 & 3\\ 3 & 2 & 1+\mu\end{pmatrix}.$$

The system has no solution if $\det \mathsf{A} = 0$ which occurs for $\mu = 18/5$. Kramer's rule gives

$$y = -\det \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 0 \\ 3 & 2 & 1+\mu \end{pmatrix} / \det \mathsf{A} = 0$$

if $\mu = 1$. Then, similarly $z = 39/(18-5\mu) = 3$ and $x = 4-3y-2z = -2$. Alternatively we reduce the system to triangular form:

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 2 & 1 & 3 & 5 \\ 3 & 2 & 1+\mu & 0 \end{array}\right) \xrightarrow[R2 \;\to\; R2-(2\times R1)]{R3 \;\to\; R3-R2-R1} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & -5 & -1 & -3 \\ 0 & -2 & \mu-4 & -9 \end{array}\right).$$

$$\xrightarrow[R3 \;\to\; R3-R2)]{R2 \;\to\; R2\times -5 \quad R3\to R3\times -2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 10 & 2 & 6 \\ 0 & 0 & 18-5\mu & 39 \end{array}\right).$$

From which we can read off the results.

5.

$$\det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{pmatrix} = 0$$

if $(1-\lambda)(-1-\lambda)(2-\lambda)+6(1+\lambda) = 0$ hence if $\lambda = -1$ or $4\lambda^2-3\lambda-4 = 0$ or $\lambda = 4$ or -1 (again). Thus we check

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} -3 & 0 & 2 \\ 0 & -5 & 0 \\ 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \\ 15 & 0 & 15 \end{pmatrix} \begin{pmatrix} -3 & 0 & 2 \\ 0 & -5 & 0 \\ 3 & 0 & -2 \end{pmatrix}$$
$$= \mathsf{0}.$$

Expanding $(\mathsf{A}+\mathsf{I})(\mathsf{A}+\mathsf{I})(\mathsf{A}-4\mathsf{I}) = 0$ gives

$$\mathsf{A}^3 - -2\mathsf{A}^2 - 7\mathsf{A} - 4\mathsf{I} = 0$$

and hence (multiplying through by A^{-1} and re-arranging)

$$4\mathsf{A} = \mathsf{A}^2 - 2\mathsf{A} - 7\mathsf{I}$$

which works out to

$$\mathsf{A}^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 3 & 0 & -1 \end{pmatrix}.$$

We can check this by showing it gives the unit matrix when multiplied by A.

Complex Numbers

1. The rule for quadratics gives

$$z = \frac{1}{2}\left[5i \pm \sqrt{-25 - 4(i-7)}\right]$$

or $z = \frac{1}{2}(5i \pm \sqrt{3-4i})$. We therefore have to find this square root in the form $a + ib$. It is (marginally) easier to take the square root of $3 + 4i$ since this is in the first quadrant so all angles are positive, and then to take the complex conjugate. To take the square root we write $3 + 4i = \sqrt{3^2 + 4^2}e^{i\psi} = 5e^{i\psi}$ where $\tan(\psi) = 4/3$. Thus, $\sin(\psi) = 4/5$ and $\cos(\psi) = 3/5$. We calculate $\cos(\psi/2)$ and $\sin(\psi/2)$ from the double angle formulae: for example, $1 - 2\sin^2(\psi/2) = \cos(\psi) = 3/5$. Finally $\sqrt{3-4i} = \sqrt{5}e^{-i\psi/2} = \sqrt{5}(\cos(\psi/2) - i\sin(\psi/2)) = 2 - i$. So for the solution of the quadratic we have $z = \frac{5}{2}i \pm (1 - i/2) = 1 + 2i$ or $-1 + 3i$. Finally $Q = (z - z_1)(z - z_2) = (z - 1 - 2i)(z + 1 - 3i)$ which can (should!) be checked by multiplying out the brackets.

3. We have

$$e^z = \exp(r\cos(\theta) + ir\sin(\theta)) = e^{r\cos(\theta)}(\cos(r\sin(\theta)) + i\sin(r\sin(\theta))).$$

So $\text{Im}(e^z) = e^{r\cos(\theta)}\sin(r\sin(\theta))$ and $\text{Re}(e^z) = e^{r\cos(\theta)}\cos(r\sin(\theta))$. Then

$$\begin{aligned}\text{Im}(e^z) &= \text{Im}(1 + z + \frac{1}{2!}z^2 + \ldots)\\ &= \text{Im}(z) + \text{Im}(z^2/2!) + \ldots\\ &= \text{Im}(re^{i\theta}) + \text{Im}(r^2e^{2i\theta}) + \ldots\\ &= r\sin(\theta) + \frac{1}{2!}r^2\sin(2\theta) + \frac{1}{3!}r^3\sin(3\theta) + \ldots.\end{aligned}$$

Thus

$$\begin{aligned}\lim_{r\to 0^+}\left[\frac{1}{r}e^{r\cos(\theta)}\sin(r\sin(\theta)) + \ldots\right] &= \lim_{r\to 0^+}\left[\sin(\theta) + \frac{1}{2!}\sin(2\theta) + \ldots\right]\\ &= \sin(\theta).\end{aligned}$$

The same result is obtained from l'Hôpital's Rule by differentiation with respect to r.

5(i) $1 + e^{i\theta} = 1 + e^{2i\phi} = 2e^{i\phi}(e^{-i\phi} + e^{i\phi})/2 = 2e^{i\phi}\cos(\phi)$.

(ii)$(1 + e^{i\theta})^n = 1 + ne^{i\theta} + \binom{n}{2}e^{2i\theta} + \ldots$. Thus

$$\begin{aligned}\text{Re}(1 + e^{i\theta})^n &= 1 + n\cos(\theta) + \binom{n}{2}\cos(2\theta) + \ldots\\ &= C_n\\ &= \text{Re}(2e^{i\phi}\cos(\phi))^n\\ &= (2\cos(\phi))^n\cos(n\phi)\end{aligned}$$

which is the required result. For $2\pi/3 < \theta < 4\pi/3$ we have $\pi/3 < \phi < 2\pi/3$ hence $0 < \cos(\phi) < 1$ and hence $(\cos(\phi))^n \to 0$ as $n \to \infty$. Also $|\cos(n\phi)| < 1$ so $C_n \to 0$ (with increasingly rapid but smaller oscillations).

7. Let $P(z) = z^3 + (2 + i)z^2 - (1 + 4i) - 2 + 3i$. Then $P(i) = 0$ by substitution and hence, by inspection, $P(z) = (z - i)(z^2 + (2 + 2i)z -$

$(3+2i)) = 0$. The solution of the quadratic is $z = 1 + i \pm \sqrt{(3+4i)}$ Then $\sqrt{(3+4i)} = 2 + i$ from problem 1. Thus, $z = 3 + 2i$ or $z = -1$.

9.

$$\begin{aligned}
\exp(xe^{i\theta}) &= \sum \frac{x^n e^{in\theta}}{n!} = \sum \left[\frac{x^n \cos(n\theta)}{n!} + i\frac{x^n \sin(n\theta)}{n!}\right] \\
&= \frac{1}{2}\left(\exp(xe^{i\theta}) + +\exp(xe^{-i\theta})\right) + \frac{i}{2i}\left(\exp(xe^{i\theta}) - \exp(xe^{-i\theta})\right) \\
&= e^{x\cos(\theta)}\cos(x\sin(\theta)) + ie^{x\cos(\theta)}\sin(x\sin(\theta)) \\
&= \left(1 + x\cos(\theta) + \frac{x^2}{2!}\cos^2(\theta) + \frac{x^3}{3!}\cos^3(\theta) + \frac{x^4}{4!}\cos^4\theta + \dots\right) \\
&\times \left(1 - \frac{x^2}{2!}\sin^2(\theta) + \frac{x^4}{4!}\sin^4(\theta) + \dots\right) + i\,(\dots)\,.
\end{aligned}$$

Comparing the real coefficients of x^4:

$$\frac{1}{4!}\cos(4\theta) = -\frac{1}{4}\cos^2(\theta)\sin^2(\theta) + \frac{1}{4!}\sin^4(\theta) + \frac{1}{4!}\cos^4(\theta)$$

or $\cos(4\theta) = -6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) + \cos^4\theta$. Considering the imaginary parts gives $\sin(4\theta) = 4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta)$.

Differential Equations

1. For a trial solution $y = e^{px}$ the auxiliary equation is $p^2 + 3p + 2 = (p+2)(p+1) = 0$ so the CF is $y_{CF} = ae^{-2x} + be^{-x}$. For a PI we write $\cos(x) = \mathrm{Re}\{e^{ix}\}$ and solve $y'' + 3y' + 2y = e^{ix}$ and take the real part. We try $y_{PI} = Ae^{ix}$ giving $-A + 3iA + 2A = 1$ or $A = (1-3i)/10$. Thus $y_{PI} = \mathrm{Re}\{(1-3i)e^{ix}/10\} = \frac{1}{10}(\cos(x) + 3\sin(x))$ and the general solution is $y = y_{CF} + y_{PI}$. Puting in the boundary condition $y(0) = 11/10$ gives $a + b = 1$ and $y'(0) = -7/10$ gives $2a + b = 1$ from which $a = 0$ and $b = 1$. Thus the solution is $y = e^{-x} + \frac{1}{10}[\cos(x) + 3\sin(x)]$.

3. For a trial solution $y = e^{px}$ the auxiliary equation is $p^2 - p - 2 = (p-2)(p+1) = 0$ so $p = 2$ or $p = -1$. The solution satisfying $y \to 0$ as $x \to +\infty$ is $y = ae^{-x}$ and then $y(0) = 1$ requires $a = 1$ so the required solution is $y = e^{-x}$.

5. For the CF we get $z_{CF} = e^t(Ae^{it} + Be^{-it})$. For the PI we try

$$\begin{aligned} z &= (a + bt)e^{2it} \\ \dot{z} &= [2i(a + bt) + b]e^{2it} \\ \ddot{z} &= \{2i[2i(a + bt) + b] + 2ib\}e^{2it}. \end{aligned}$$

Substituting and comparing coefficients of powers of t we get

$$(-2 + 4i)a + (-2 + 4i)b = 2 \quad \text{and} \quad b(1 + 2i)t = 5t,$$

hence $a = -2$ and $b = (1 - 2i)$. The general solution is therefore

$$z = e^t(Ae^{it} + Be^{-it}) + [(1 - 2i) - 2]e^{2it}.$$

The boundary condition $z(0) = 0$ gives $A+B = -2$ and $\dot{z}(0) = 0$ gives $i(A-B) = -1+6i-(A+B)$ from which $A = -\frac{1}{2}-2i$ and $B = -\frac{5}{2}+2i$.

7. We solve $y'' + 2y' + 2y = \text{Im}\{e^{2ix}\}$ and take the imaginary part of the solution. The solutions of the homogeneous equation are $e^{(-1\pm i)x}$ so the CF can be written as $y_{CF} = e^x[a\cos(x) + b\sin(x)]$. For a PI we try $y = Ae^{2ix}$ giving $A = -\frac{1}{10}(1 + 2i)$. Then $\text{Im}\{-\frac{1}{10}(1 + 2i)e^{2ix}\} = -\frac{1}{10}\sin(2x) - \frac{1}{5}\cos(2x)$. The general solution is

$$y = e^{-x}[\cos(x) + b\sin(x)] - \frac{1}{10}\sin(2x) - \frac{1}{5}\cos(2x)$$

and the boundary conditions give, from $y(0) = 0$, $a = 1/5$ and from $y(\pi/2) = 0$, $b = -\frac{1}{5}e^{\pi/2}$.

9. The general solution is $y = ae^{3t} + be^{-4t} - \frac{1}{12}e^{-t}$. This remains finite as $t \to \infty$ if $a = 0$. Then, $y(0) = a + b - \frac{1}{12}$ and $y'(0) = 3a - 4b + \frac{1}{12}$. Putting $a = 0$ and eliminating b gives the required

relation:
$4y(0) + y'(0) = -1/4.$

11. (i) We have

$$\int \frac{dy}{\sqrt{(1-y^2)}} = \sin^{-1}(y) = x^4/4 + \phi$$

ϕ an arbitrary constant, and hence $y = \sin(x^4/4 + \phi)$.

(ii) The integrating factor is $\exp\left(\int \frac{\cos(x)}{\sin(x)}\right) dx = \exp\left(\int \frac{d\sin(x)}{\sin(x)}\right)$ $= \exp[\ln(\sin(x))] = \sin(x)$ so the equation can be written as

$$\frac{1}{\sin(x)}\,(y\sin(x))' = 2\cos(x)$$

from which $y\sin(x) = -\sin^2(x) + c$ or $y = -\sin(x) + c/(\sin(x))$.

(iii)$y = ae^{3x} + be^{x}$, with a and b arbitrary constants.

(iv) The auxiliary equation has the repeated root $p = 2$ so the solution is of the form $y = (a + bt)e^{2x}$ with a and b arbitrary constants.

13. (i) We write this as $y'/y = 3x^2$ which integrates to $y = A\exp(x^3)$.

(ii) Multiplying through by $\sin(x)$ the equation becomes $(\cos(x)y)' = 2x$ which integrates to $y = (x^2 + c)/\cos(x)$.

(iii) Since the auxiliary equation has 5 as a repeated root the solution is $y = (a + bx)e^{5x}$.

(iv) Taking $y = Ae^{x}$ as the trial solution for a PI gives $A = 1$. The general solution is $y = ae^{4x} + be^{3x} + e^{x}$.

Multiple Integrals

1. $dV = r^2 \sin(\theta) d\theta d\phi$.
We have

$$\begin{aligned}
I &= \int_0^{2\pi} \int_0^{\infty} \int_0^{\pi} \frac{1}{r} \exp[ir\cos(\theta) - \epsilon r] r^2 d(-\cos(\theta)) dr d\phi \\
&= -2\pi \int_0^{\infty} \left[\frac{1}{i} \exp(ir\cos(\theta)) \right]_1^{-1} e^{-\epsilon r} dr \\
&= 2\pi i \int_0^{\infty} \left[e^{-ir-\epsilon r} - e^{ir-\epsilon r} \right] dr \\
&= 2\pi i \left[\frac{e^{-ir-\epsilon r}}{i+\epsilon} - \frac{e^{ir-\epsilon r}}{i-\epsilon} \right]_0^{\infty} \\
&= 2\pi i \left[-\frac{1}{i+\epsilon} + \frac{1}{i-\epsilon} \right] \\
&= \frac{4\pi}{1+\epsilon^2}
\end{aligned}$$

from which the result follows by taking the limit $\epsilon \to 0$.

3. The element of solid angle on the surface is $d\Omega = dA\cos(i)/r^2$ where $dA = ad\phi dz$, $r = (a^2 + z^2)^{1/2}$ and the angle between the radial vector from the origin and the normal at a point on the surface is given by $\cos(i) = a/(a^2 + z^2)^{1/2}$. It remains to figure out the limits of z and ϕ. The limits for z are $\pm x$ i.e. $\pm a\cos(\phi)$. Since x is positive ϕ must lie between $-\pi/2$ and $+\pi/2$. Putting this together gives the required result.

Partial Derivatives

1. $x = r\cos(\theta)$, $y = r\sin(\theta)$ implies $r^2 = x^2 + y^2$ so $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$. Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.
If $g(r,\theta) = f(r)\sin(\theta) = yF(r)$ then
(i) $\dfrac{\partial g}{\partial x} = y\dfrac{dF}{dr}\dfrac{\partial r}{\partial x} = \dfrac{xy}{r}\dfrac{dF}{dr}$
(ii) $\dfrac{\partial g}{\partial y} = F + \dfrac{y^2}{r}\dfrac{dF}{dr}$
(iii)

$$\begin{aligned}\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} &= \left[\frac{y}{r} - \frac{xy}{r^2}\frac{x}{r} + \frac{y}{r} - \frac{y^3}{r^3} + \frac{2y}{r}\right]\frac{dF}{dr} + \left[\frac{xy}{r}\frac{x}{r} + \frac{y^3}{r^2}\right]\frac{d^2F}{dr^2}\\ &= y\left(\frac{d^2F}{dr^2} + \frac{3}{r}\frac{dF}{dr}\right)\\ &= 0\end{aligned}$$

if

$$F'' + \frac{3}{r}F' = \frac{1}{r^3}\left(r^3F'\right)' = 0$$

which integrates to $F = B/r^2 + A$ and hence $f(r) = B/r + Ar$ where A and B are constants.

3. $f(x,y) = x^4 + 2x^2 + 3xy + 3y$ so $f_y = 3x + 3 = 0$ if $x = -3$ and $f_x = 4x^3 + 4x + 3y = 0$ if $y = 40$. So the stationary point is $(-3, 40)$ and $f(-3, 40) = -141$. Then $f_{xx}(-3, 40) = 12x^2 + 4 = 112$, $f_{xy}(-3, 40) = 3$, and $f_{yy}(-3, 40) = 0$ so the Taylor series is $f(x,y) = -141 + 56(x+3)^2 + 3(x+3)(y-40) = -141 + (x-3)[56(x+3) + 3(y-40)]$. Thus, putting for example $x = -3 + 3/56$ we find $f(x,y) > f(-3, 40)$ if $y > 40$ and $f(x,y) < f(-3, 40)$ if $y < 39$. Thus the stationary point is neither a maximum nor a minimum.

5. $f(x,y) = x^4 - 2x^2 + y^3 - 3y$ so $f_x = 4x^3 - 4x = 0$ if $x = 0$ or$x = \pm 1$ and $f_y = 3y^2 - 3 = 0$ if $y = \pm 1$. Then $f_{xx} = 12x^2 - 4$, $f_{xy} = 0$ and

$f_{yy} = 6y$. Thus, if $\Delta(x, y) = f_{xx} fyy - fxy^2$, for the six stationary points we have

(a) $\Delta(0, 1) = -24$, $f_{xx} < 0$, maximum

(b) $\Delta(0, -1) = 24$, saddle

(c) $\Delta(1, 1) = 48$, saddle

(d) $\Delta(1, -1) = -48$, $f_{xx} > 0$, minimum

(e) $\Delta(-1, 1) = 48$, saddle

(f) $\Delta(-1, -1) = -48$, $f_{xx} > 0$, minimum

Near $(1, 1)$, $f(1.1, 1.1) \approx -3 + 4(x-1)^2 + 3(y-1)^2 = -3 + 7 \times 10^{-2}$ so the percentage error is $7 \times 10^{-2}/3 \approx 2.3\%$. Near $(2, 2)$ we can use the binomial expansion to find the linear terms in the Taylor series (the quadratic ones will be smaller) so $f(2 + \delta, 2 + \epsilon) \approx 16(1 + 2\delta) - 8(1 + \delta) + 8(+3\epsilon/2 - 6(1 + \epsilon/2) = 10 + 24\delta + 9\epsilon$. Thus we take $\delta = +0.2$ and $\epsilon = +0.2$ for the maximum error giving $f \approx 16.6$ so the percentage error is $6.6/10$ or 66%. The function is slowly varying around a stationary point, so a small error in the arguments gives only a small (second order) error in the function. Conversely, near a stationary point a very accurate measurement of the function is required to fix the arguments.

7. $f(x, y) = xy + 1/x + 1/y$ so $f_x = y - 1/x^2 = 0$ if $y = 1/x^2$ and $f_y = x - 1/y^2 = 0$ if $x = 1/y^2 = x^4$ (provided $x \neq 0$). The only solution is $(1, 1)$. Then $f_{xx}(1, 1) = 2/x^3 = 2$ and $f_{xy} = 1$, $f_{yy} = 2$. Thus, $f_{xx} fyy - f_{xy}^2 = 3 > 0$ at $(1, 1)$ so this is a saddle point.

9. $f(x, y) = \ln(x) - x/y^2 - 2y$ so $f_x = 1/x - 1/y^2 = 0$ if $x = y^2$, (provided $x \neq 0$) and $f_y = -2x/y^3 - 2 = 0$ if $x = -y^3$. Thus, we require $y^2 = -y^3$ which gives $(1, -1)$ as the only solution. Then $f_{xx}(1, -1) = -1/x^2 = -1$, $f_{xy}(1, -1) = -2/y^3 = 2$ and $f_{yy}(1, -1) = 6x/y^4 = 6$. Thus $f_{xx} f_{yy} - f_{xy}^2 = -10 < 0$ at $(1, -1)$ and $f_{xx} < 0$ so this is a maximum.

Partial Differential Equations

1.
$$y(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$

Hence $y(x,0) = F(x) + \frac{1}{2c}\int_x^x G(s)ds = F(x)$ and $\partial y/\partial t(x,0) = -\frac{c}{2}F'(x) + \frac{c}{2}F'(x) + \frac{1}{2c}[cG(x) - (-c)G(x)] = G(x)$. That this is a solution of the wave equation can be verified by direct differentiation using $\partial F(x-ct)/\partial t = -c\partial F(x-ct)\partial x = cF'$ etc.

3. Near the origin the waveform is $\frac{1}{2}a(l+(x-ct)) + \frac{1}{2}a(l-(x+ct))$. At $x = 0$ both contributions vanish for $t = l/c$. The parts of the waveform at rest for $t > l/c$ are $x < -ct-l$, $x > ct+l$ and $-ct+l < x < ct-l$. It is perhaps easiet to see this from the picture, rather than algebraically, by noting that the evolution consists of one half of the triangular waveform moving to the right and the other half moving to the left.

5. Seeking a solution of the form $y(x,t) = X(x)T(t)$ we deduce that $\ddot{T} = -\omega^2$ and $X'' = -\omega^2/c^2$ hence that y is a sum of products of $\cos(\omega t)$ or $\sin(\omega t)$ with $\cos(\omega x/c)$ or $\sin(\omega x/c)$. The condition $y(x,0) = B\sin(\pi x/l)$ requires $\omega = c\pi/l$ and $y(x,t) = \sin(\pi x/l)(A\sin(\pi ct/l) + B\cos(\pi ct/l)$. Finally, $\partial y/\partial t(x,0) = v\sin(\pi x/l)$ gives $A = lv/\pi c$.

Evaluating $E = \dot{y}^2 + (\pi c/l)^2 y^2$ with $y = \sin(\pi x/l)(B\cos(\pi ct/l) + (lv/\pi c)\sin(\pi ct/l))$ gives $E = \sin^2(\pi x/l)(v^2 + B^2)(\pi c/l)^2$ which is constant in time at each value of x.

7. The separable solution satisfying the conditions at $x = 0$ and $x = l$ is $y(x,t) = \sum_n (a_n \sin(n\pi ct/l) + b_n\cos(n\pi ct/l))\sin(n\pi x/l)$. To satyisfy $\partial y/\partial t(x,0) = 0$ we set $a_n = 0$ and to satisfy $y(x,0) = a\sin(\pi x/l) + b\sin(3\pi x/l)$ we must have

$$y(x,t) = a\cos(\pi ct/l)\sin(\pi x/l) + b\cos(3\pi ct/l)\sin(3\pi x/l).$$

9. The general solution gives

$$\begin{aligned} y(x,t) &= \frac{1}{2}e^{-(x-ct)^2} + \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2c}\int_{x-ct}^{x+ct} cxe^{-x^2}\,dx \\ &= \frac{1}{2}e^{-(x-ct)^2} + \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{4}\left[-e^{-x^2}\right]_{x-ct}^{x+ct} \\ &= \frac{3}{4}e^{-(x-ct)^2} + \frac{1}{4}e^{-(x+ct)^2} \end{aligned}$$

as required. After a sufficiently long time the maximum displacement of the string will be $\frac{3}{4}$ (the peak moving to the right). Thus the maximum will be at $x = 0$ as long as the displacement at $x = 0$ is greater than $\frac{3}{4}$, hence for a time given by $y(0,t) = \frac{3}{4}e^{-(ct)^2} + \frac{1}{4}e^{-(ct)^2} = e^{-(ct)^2} > 3/4$ or $t < \frac{1}{c}\left[\ln(\frac{4}{3})\right]^{1/2}$.

Fourier Series

1. $f(\theta) = \theta^2(-\pi \le \theta < \pi)$ is an even function so we seek a series representation $f(\theta) = \frac{1}{2}a_0 + \sum_n a_n \cos(n\theta)$. We have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2 d\theta = 2\pi^2/3 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2 \cos(n\theta) d\theta \\ &= \frac{1}{\pi}\left[\frac{\sin(n\theta)}{n}\theta^2\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} \theta \sin(n\theta) d\theta \\ &= \frac{-2}{n\pi}\left[-\theta\frac{\cos(n\theta)}{n}\right]_{-\pi}^{\pi} + \frac{2}{\pi n^2}\int_{-\pi}^{\pi} \cos(n\theta) d\theta \\ &= \frac{-2}{n^2\pi}\left[-\pi\cos(n\pi) + \pi\cos(-n\pi)\right] \\ &= \frac{4}{n^2}\cos(n\pi) = \frac{4}{n^2}(-1)^n \end{aligned}$$

Thus $\theta^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos(n\theta)$.

Putting $\theta = 0$ gives $\frac{\pi^2}{12} = \sum \frac{(-1)^{n+1}}{n^2}$.

3. We write $\sin(\pi x) = \frac{1}{2}a_0 + \sum a_n \cos(n\pi x)$ with

$$
\begin{aligned}
a_0 &= 2\int_0^1 \sin(\pi x)dx = \frac{2}{\pi}\left[-\cos(\pi x)\right]_0^1 = 4/\pi \\
a_n &= 2\int_0^1 \sin(\pi x)\cos(n\pi x)dx = \int_0^1 \left[\sin((n+1)\pi x) - \sin((n-1)\pi x)\right]dx \\
&= \frac{1}{\pi(n+1)}\left[\cos((n+1)\pi x)\right]_0^1 - \frac{1}{\pi(n-1)}\left[\cos((n-1)\pi x)\right]_0^1 \\
&= \frac{1}{\pi(n+1)}\left[\cos((n+1)\pi) - 1\right] - \frac{1}{\pi(n-1)}\left[\cos((n-1)\pi x) - 1\right] \\
&= 0 \quad n \text{ odd.}
\end{aligned}
$$

For n even, $n = 2p$ say, we get $a_n = -\frac{2}{\pi}\left[\frac{1}{2p+1} - \frac{1}{2p-1}\right] = \frac{4}{\pi}\frac{1}{4p^2-1}$.

Thus $\sin(\pi x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{p=1}^\infty \frac{\cos(2p\pi x)}{4p^2-1}$.

Putting $x = 1/2$ gives $\sum_{p=1}^\infty \frac{(-1)^p}{4p^2-1} = -\frac{\pi}{4} + \frac{1}{2}$.

5. We can write the function as a sum of an even and odd functions $f(\theta) = f_+(\theta) + f_-(\theta)$ as

$$
f(\theta) = \begin{cases} \frac{1}{2}(\alpha+\beta) + \frac{1}{2}(\alpha-\beta) & \text{for } -\pi leq \theta < 0 \\ \frac{1}{2}(\alpha+\beta) - \frac{1}{2}(\alpha-\beta) & \text{for } 0 \leq \theta < \pi \end{cases}
$$

The symmetric part of f $((\alpha+\beta)/2)$ contributes only to the cosine terms and the antisymmetric part to only the sine terms. Futhermore, the Fourier series of a constant is a contant so only a_0 survives in the cosine terms and we have immediately

$$f(\theta) = \frac{1}{2}(\alpha+\beta) + \sum b_n \sin(n\theta)$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi \frac{1}{2}(\alpha-\beta)\sin(n\theta)d\theta = \frac{\alpha-\beta}{n\pi}\left[-\cos(n\theta)\right]_0^\pi = \begin{cases} \frac{2(\alpha-\beta)}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thus $f(\theta) = \frac{1}{2}(\alpha+\beta) + \frac{2(\alpha-\beta)}{\pi}\sum \frac{(-1)^{n+1}}{2n+1}\sin((2n+1)\theta)$.
Putting $\theta = \pi/2$ gives $\beta = \frac{1}{2}(\alpha+\beta) - \frac{2(\alpha-\beta)}{\pi}\sum \frac{(-1)^n}{2n+1}$, which we can re-arrange to $\pi/4 = \sum \frac{(-1)^n}{2n+1}$.

7. We obtained the Fourier series for x^2 in problem 1. Since the Fourier series of a sum is the sum of the respective series, we need only find the series for x and add it to that for x^2. We have (since x is antisymmetric) $x = \sum b_n \sin(nx)$ where $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)dx$. We can evaluate this by integration by parts or as follows:

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(-\frac{\partial}{\partial\lambda}\right)\int_{-\pi}^{\pi}\cos(\lambda x)dx\bigg|_{\lambda=n} \\ &= \frac{1}{\pi}\left(-\frac{\partial}{\partial\lambda}\right)\left[\frac{\sin(\lambda x)}{\lambda}\right]_{-\pi}^{\pi}\bigg|_{\lambda=n} \\ &= \frac{1}{\pi}\left(-\frac{\partial}{\partial\lambda}\right)\left[\frac{\sin(\lambda\pi)}{\lambda}\right]_{\lambda=n} \\ &= \frac{2}{\pi}\left[-\frac{\pi\cos(\lambda\pi)}{\lambda} + \frac{\sin(\pi\lambda)}{\lambda^2}\right]_{\lambda=n} \\ &= \frac{2}{n}\left[-\cos(n\pi)\right] = \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

9. The inverse transform is

$$\int_{-\infty}^{\infty} e^{ikx}e^{-k^2}dk = \int_{-\infty}^{\infty} \exp\left(k + \frac{ix}{2}\right)^2 e^{-x^2/4}dk$$
$$= e^{-x^2/4}\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}e^{-x^2/4}.$$

Vector Calculus

1. $\oint(\mathbf{A}\times\mathbf{r})\cdot d\mathbf{r} = \int_S \nabla\times(\mathbf{A}\times\mathbf{r})\cdot d\mathbf{S}$ by Stokes' theorem.
Now, there are several ways of working out $\nabla\times(\mathbf{A}\times\mathbf{r})$. The easiest is to take $\mathbf{A}$ in the z-direction, so $\mathbf{A} = (0,0,A)$, and the origin at the centre of the circle, so $\mathbf{r} = (x,y,0)$. Then $\mathbf{A}\times\mathbf{r} = (-Ay, Ax, 0)$ and $\nabla\times(-Ay,Ax,0) = (0,0,2A)$. Finally $\displaystyle\int_S 2AdS = 2A\pi$, where S is the unit disc, hence has area π. Alternatively, using the identity, $\nabla\times(\mathbf{A}\times\mathbf{r}) = \mathbf{r}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{r} + \mathbf{A}\nabla\cdot\mathbf{r} - \mathbf{r}\nabla\cdot\mathbf{A}$ we are left with $\nabla\times(\mathbf{A}\times\mathbf{r}) = -\mathbf{A}\cdot\nabla\mathbf{r} + \mathbf{A}\nabla\cdot\mathbf{r} = -\mathbf{A}+3\mathbf{A} = 2\mathbf{A}$ and $2\mathbf{A}\cdot\mathbf{n} = 2A$ since $\mathbf{A}$ and $\mathbf{n}$ are parallel.

We have

$$\frac{1}{2}\oint(\mathbf{A}\times\mathbf{r})\cdot d\mathbf{r} = \frac{1}{2}\oint \mathbf{A}\cdot(\mathbf{r}\times d\mathbf{r}) = \frac{1}{2}\mathbf{A}\cdot\oint \mathbf{r}\times d\mathbf{r} = \pi A.$$

Thus $\displaystyle\frac{1}{2}\oint \mathbf{r}\times d\mathbf{r} = \pi =$ area of C.

3. $\displaystyle\int_V \nabla\cdot(r^2\mathbf{r})dV = \int_S r^2\mathbf{r}d\mathbf{S}$. Since the integrand is symmetrical in x, y and z the value of the intergral is the same on each face. Thus,

taking the face $x = 1/2$:

$$\begin{aligned}\int r^2\mathbf{r}\cdot d\mathbf{S} &= 6\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\frac{1}{4}+y^2+z^2)(\frac{1}{2},y,z)\cdot(dydz,0,0)\\ &= 3\left[\frac{y}{4}+\frac{y^3}{3}+z^2y\right]_{-1/2}^{+1/2}dz\\ &= 3\int(\frac{1}{3}+z^2)dz\\ &= \frac{5}{4}.\end{aligned}$$

This can be checked by calculating the integral directly as a triple integral using the result that $\nabla\cdot(r^2\mathbf{r}) = 5r^2 = 5(x^2+y^2+z^2)$.

5. $\frac{1}{3}\int\mathbf{r}\cdot d\mathbf{S} = \frac{1}{3}\int\nabla\cdot\mathbf{r}dV = \frac{1}{3}\int 3dV = V$. For the face $x = a/2$ of the cube,

$$\begin{aligned}\int\mathbf{r}\cdot d\mathbf{S} &= \int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\left(x^2+y^2+\frac{a^2}{4}\right)^{1/2}\frac{a}{2}\frac{dxdy}{\left(x^2+y^2+\frac{a^2}{4}\right)^{1/2}}\\ &= \frac{a}{2}\times a\times a = \frac{a^3}{2}.\end{aligned}$$

Summing over the 6 faces gives $3a^3$ which is 3 times the volume as required.